TJUSAMO 2011 - Invariants/Monovariants

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1 Invariants

Often, an Olympiad problem will ask you to consider a process which is used to modify some object. In this case, you should consider finding an invariant - a property of the object which is not changed by the process. This invariant might be useful for proving that an object can never be reached through (repeated) application of the process. (It also might not - for example, the number 1 is always an invariant, but the usefulness of that fact is limited.) The simplest and often most useful invariants are numbers taken mod 2.

2 Monovariants

Invariants aren't always easy to find. Instead, one might try to find monovariants. A (strict) monovariant is a quantity which always (strictly) increases or always (strictly) decreases. Note that the *existence* of a bounded strict monovariant which is always an integer implies immediately that the process must halt. Invariants and monovariants can also be used to investigate processes which are not explicitly defined in the problem.

3 Problems

1. The numbers $1, 2, \cdots, 2010$ are written on the board. It is permitted to take two numbers a, b and replace them with $a - b$. Prove that if this operation is applied until there is only one number on the board, this number is odd.

2. The ordered pairs of numbers $(0,1), (1,2), (2,3), \ldots, (2010, 2011)$ are written on a blackboard. You can take any two pairs of numbers (a, b) and (c, d) and replace them with the pair $(ac - bd, ad + bc)$. You do this until only one pair is left. What are all possible values for this pair?

3. In a cube, there are seven vertices marked 0 and one marked 1. It is permitted to add 1 to any two neighboring vertices (that is, two vertices connected by an edge). Is it possible that all the numbers are divisible by 3 after a finite number of steps?

4. Prove that every positive rational number can be written in the form $a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots \frac{1}{a_n}}}$, for some positive integers n and $a_1, a_2, \cdots, a_n$.

5. The numbers in $1, 2, \cdots, n$ are written in a line. It is permitted to swap the positions of any two numbers. After this operation is applied repeatedly, the numbers are all in their original positions. Prove that the operation was applied an even number of times.

4 More Problems

6. In an $n \times n$ grid of numbers from $\{-1, 1\}$, there is one -1 in each row and each column. It is permitted to negate all the numbers in any given row or column. Find the minimum possible number of -1's in a grid which can be reached by these operations.

7. Several positive integer numbers are written on a blackboard. One can erase any two distinct integers and write their greatest common divisor and least common multiple instead.

   (a) Prove that eventually the numbers will stop changing.

   (b) Prove that the values of the numbers, once they stop changing, do not depend on what moves were made.

8. Four congruent right triangles are given. One can cut one of them along the altitude and repeat the operation several times with the newly obtained triangles. Prove that no matter how we perform the cuts, we can always find among the triangles two that are congruent.

9. Let n be a positive integer. Define a sequence by setting $a_1 = n$ and, for each $k > 1$, letting a_k be the unique integer with $0 \le a_k \le k - 1$ for which $a_1 + a_2 + \cdots + a_k$ is divisible by k. For instance, when $n = 9$ the obtained sequence is $9, 1, 2, 0, 3, 3, 3, \ldots$. Prove that for every n the sequence $a_1, a_2, \ldots$ eventually becomes constant.

10. Let p_n be the nth prime. Let $0 < r < 1$ be a real. Define the sequence x_n by $x_0 = r, x_n = \dfrac{p_{n-1}}{x_{n-1}} - \left\lfloor \dfrac{p_{n-1}}{x_{n-1}} \right\rfloor$ if $x_{n-1} \neq 0$, and $x_n = 0$ if $x_{n-1} = 0$. Find all r for which the sequence is eventually constant.

11. Three nonnegative real numbers r_1, r_2, r_3 are written on a blackboard. These numbers have the property that there exist integers a_1, a_2, a_3, not all zero, satisfying $a_1 r_1 + a_2 r_2 + a_3 r_3 = 0$. We are permitted to perform the following operation: find two numbers x, y on the blackboard with $x \le y$, then erase y and write $y - x$ in its place. Prove that after a finite number of such operations, we can end up with at least one 0 on the blackboard.