Complex 2 - D Vector Space Arithmetic

You will be learning the framework of Quantum Mechanics in the first semester of the Electrodynamics and Quantum course this coming year. The approach will be to generalize from the simplest kind of real world quantum system, a so-called 2 state system also known an a Quantum Bit (QuBit) . The math of Qubits contains most of the abstract mathematical structures used in building quantum models of more complicated systems. During the school year we will rapidly advance to describing the abstract linear algebra structure of the general framework so **every student in the class needs to get used to the arithmetic of the building blocks of the QuBit case over the summer**. The relatively easy and concrete cases you meet in 2-D will help you build intuition for the general abstract linear algebra structure that is the framework of Quantum Mechanics in any number of dimensions. Our mathematical universe in these notes is the *matrix arithmetic* of complex numbers, 2 dimensional complex vectors, and 2 x 2 complex matrices.

This universe is populated by FOUR kinds of mathematical objects : **complex numbers, 2-component complex vectors (we will call them "ket"s in this context), dual vectors that are also 2-component complex vectors (we will call them 'bra's in this context), and 2 x2 matrices of complex numbers that we will call 'operators' in this context**. You will be learning what was once called the (2 dimensional) 'Matrix Mechanics'. This note provides a basic description of this arithmetic and references where you can learn more about it. The wording in this note is not 'math minimalist' and some things will be repeated for emphasis because I assume that you are reading about these things for the first time. To help you get ready to do physics with this matrix arithmetic (usually called a matrix *algebra* because it involves notions of addition and multiplication) , *mandatory* practice is provided at various points. If you master this arithmetic you will be in good shape to hit the ground running in September.

Once you are confident with the arithmetic, you will be able to understand the **algebraic abstraction layers as the properties of concrete quantitative objects**, not just as some high minded abstract system. In these notes I will also introduce a notation for vectors created by Paul Dirac called 'bra-ket' notation. (Take a minute to google Paul Dirac; he is the father of the modern formulation of Quantum Mechanics. If you are deeply interested in this subject then eventually you must read Dirac's book, *The Principles of Quantum Mechanics. This is where I learned most of what I will be teaching you in the way of basic formalism of the theory.*).

How to Do this Assignment:

1) ***Do the Complex Numbers part completely first***. All of the rest of the matrix arithmetic assumes you know all about complex numbers. The scalar components of all vectors and matrices you meet after that are generally complex numbers (not real numbers). So get complex arithmetic down cold first.
2) *After your are double sure that Complex numbers are your closest friends*, read the sections of these notes on kets, bras, and operators. There are a few references to other articles. Read them as you need to , especially when these notes don't make good sense to you or you are left feeling that you need more context to understand the basic ideas. We will have much more to say about the overall abstract algebraic structure of these kinds of quantities as the course progresses so please concentrate on mastering the matrix manipulations at this stage. There is a multiplication table, memorize it.
3) Work through all the problems by hand (you might use *Mathematica* or Wolfram Alpha to check your calculations, but DO THEM IN DETAIL BY HAND).

4) If you find these notes too cryptic then take the responsibility to read some of the other references provided and work on it until ALL the problems are completely clear to you. Feel free to discuss the problems with whomever you like, but **be prepared for a test on this material at the level of the problems during the first week of class**. Solutions to the problems will be posted one week prior to the beginning of school.

Complex Numbers

Complex numbers are fundamental to everything that we will do in quantum mechanics so you need to be totally familiar with their basic properties and facile with their computational manipulations. Complex numbers are the 'ultimate numbers' in the sense that they provide the largest extension that contains the real numbers and preserves the basic algebraic properties of addition, multiplication, and the relationships between them. Complex numbers make it possible to solve all equations in one variable that arise from finite additions and multiplications (i.e. polynomial equations). There are many good review articles on complex numbers. I have chosen two in particular that I would like you to read and, happily, they come with some problems you can use to confirm your understanding so I have embedded these articles in the problem sets found below **Three of the four problem sets below are mandatory and *problems of these types will appear on the first week test*.**

Review Articles and Problem Complex Number Problem Sets

Complex Numbers Problem Set Part 1: Do all 10 problems at the end of the short review article:
http://www.math.columbia.edu/~rf/complex.pdf

Complex Numbers Problem Set Part 2: Do all 14 problems at the end of the short review article:
https://www.chem.tamu.edu/rgroup/hughbanks/courses/673/handouts/complex_numbers.pdf

Complex Numbers Problem Set Part 3: Thinking of Complex Numbers as Vectors in a plane.
If we represent a pair of good-old two component real vectors in the plane as a pair of complex numbers (w and z, say), then how do we write the standard euclidean inner (the so-called 'dot' product) product of the two vectors in terms of the complex numbers w and z ? There is also a way to write something that is closely related to the vector cross product. How do we write that expression in terms of the complex numbers w and z ?

Complex Numbers Problem Set Part 4: This problem is optional (but might appear as an extra credit problem on the first week test). No solution to this problem will be posted :)

Complex numbers can be used to represent points on a plane. Given three points in the plane with representations as complex numbers p1,p2, and p3 respectively, these points determine a circle if they are not collinear. This circle has a center that can also be represented as a complex number; lets call it c.

The problem:
Derive an algebraic formula for c in terms of p1,p2, and p3 (you are allowed to use p1,p2, p3, and their complex conjugates but otherwise the formula must use only algebraic operations,i.e. operations of complex addition, subtraction, multiplication, and division).
The best logic can have holes so please humor me and verify that your formula works for some easy cases and also that it fails to yield an answer for c when p1,p2, and p3 are collinear.

Additional Learning/Review Resources for Complex Numbers

If you find the review articles provided above and associated problems too abrupt a review and need to start more gently or perhaps you simply want to learn more, then the following resources might help:

Start Up Worksheet Approach: A step by step tutorial for basic operations with complex numbers if you remember very little about manipulating complex numbers and benefit from a work-sheet kind of review of the basics (with solutions of embedded exercises):

http://www.wtamu.edu/academic/anns/mps/math/mathlab/col_algebra/col_alg_tut12_complexnum.htm

If you are a Math Team Kid : and want the ALL ENCOMPASSING book on Complex Numbers that will also tie in complex numbers to a lot of other math that you know, written by a legend in the IMO world, then read Complex Numbers from A to Z by T. Andreescu and D. Andrica .

Another **highly mathy but complete treatment of complex numbers is here**: http://www.numbertheory.org/-book/cha5.pdf

Wikipedia: While you can use the Wikipedia as a reference, the Complex Numbers article is rather incoherent for my taste as a review or introduction. But its all there.

Khan Academy: As with most Khan Academy lectures, the complex numbers lectures are a bit mushy/imprecise in their language (I assume because Khan is trying to communicate ideas for the first time in 15 minute videos he feels that its OK to be a little vague on the first pass.) Watch these if you remember nothing about complex numbers and are starting from scratch.

For this students who want to go Internet searching to review and learn more about the math of Complex numbers, the following search terms will prove useful *and its all terminology you need to know*: complex number, imaginary unit, Cartesian form of complex numbers, real part (of complex number), imaginary part (of a complex number),polar form of complex numbers, modulus of complex number, magnitude of complex number, phase of complex number, uni modular complex number, complex conjugate, addition and subtraction of complex numbers, multiplication and division of complex numbers, complex plane, the fundamental theorem of algebra, Euler’s formula, de Moivre’s theorem, (complex) roots of unity, complex numbers as two component real vectors, trigonometric identities and complex numbers, principal value. (While beyond the scope of a review of the basic properties of complex numbers and rational functions of a complex variable themselves, additional terminology is used to describe interesting classes of functions of a complex variable. Prominent in this area are the terms Cauchy-Riemann equations, holomorphic and meromorphic functions.)

Ket Vectors

The notation for a ket vector is | *name* > ; where *name* is some unique name/ symbol / or label for the vector. This notation was invented by Paul Dirac and is generally known as bra-ket notation.

Some other learning materials about bra – ket notation :

Dirac' s Book, *Priniciples of Quantum Mechanics*, for the full story

And then there is
http : // en.wikipedia.org / wiki / Bra–ket_notation

http : // www.physics.unlv.edu / ~bernard / phy721_ 99 / tex_notes / node6.html

More mathemtical depth :

http : // quantum.phys.cmu.edu / CQT / chaps / cqt03.pdf or

http : // ocw.mit.edu / courses / electrical – engineering – and – computer – science / 6 – 974 – fundamentals – of – photonics – quantum – electronics – spring – 2006 / lecture – notes / chapter5.pdf

In our concrete arithmetic we can represent every ket vector by a column of 2 complex numbers :
|v> := $\begin{pmatrix} v^1 \\ v^2 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$; where a and b are complex numbers. The complex numbers a and b are called the scalar components of |v>. Sometimes we will label the complex numbers with indices (e.g. the 1 and 2 attached as superscripts to the symbol v) or refer to the collection of two numbers as v^i ; where we understand that the index i runs from 1 to 2. This ‘index notation’ will become useful when we move on to vectors with many components and for manipulating more complicated expressions involving vectors of all kinds, but in these notes I am going to stay away from this more abstract notation where we refer to collections of numbers in this compact form because I want you to ***visualize*** **the basic manipulations of these collections as matrix operations on structured lists of complex numbers** and learn to do them ‘by hand’. *Two operations that yield another ket are multiplication by a complex number and addition of kets.*

Multiplication of a ket by a complex number

If α is a complex number and |v> is the ket above, then α |v > = $\alpha \begin{pmatrix} a \\ b \end{pmatrix} := \begin{pmatrix} \alpha\, a \\ \alpha\, b \end{pmatrix}$.

I am using the := notation to mean that the thing on the side with the : is defined by the expression on the other side, so you would read ‘a:=2’ as ‘a is defined to be 2’. So this defines what we mean by multiplying by a scalar.

Addition of kets

If |v>:= $\begin{pmatrix} v^1 \\ v^2 \end{pmatrix}$ and |w>:= $\begin{pmatrix} w^1 \\ w^2 \end{pmatrix}$ then the sum of these two kets (what + means in this context) is

|v> + |w> := $\begin{pmatrix} v^1 + w^1 \\ v^2 + w^2 \end{pmatrix}$;

so addition is just “component by component” addition of the two columns of *complex* numbers to make another column of numbers. Relative to addition of kets, there is a unique zero ket |0> := $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ **.** If we add the zero ket to any ket |v> then the result is just |v>. By the properties of complex numbers, we get the zero ket if we multiply any ket by (the complex number) 0.

Linear Combinations

We often combine the operations of multiplication by complex numbers and addition of kets to make *linear combinations* of two or more kets where we add together multiples of kets with complex coefficients. For two kets we make the linear combination with complex number coefficients α and β like so:

$$\alpha\,|v> + \beta\,|w> = \begin{pmatrix} \alpha v^1 + \beta w^1 \\ \alpha v^2 + \beta w^2 \end{pmatrix}$$

You should notice that for **2** dimensional vectors like these, any ket at all can be expressed as a unique linear combination of the **2** vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ **.** There are lots of pairs of vectors with this property (actually a four fold complex infinity of them) ; for instance, its easy to show that any vector can be expressed as a complex linear combination of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.[*At points in these notes where I say ‘its easy to show’ then you should make sure that you can do it! I am not being lazy by not explaining, instead I want you to do it as part of your active reading of this text.*]

A little more terminology: When every vector in a vector space can be expressed as a linear combination of a collection of vectors we say that the vectors in the collection **span** the vector space. The minimum number of vectors needed to span a space is usually taken as a formal definition for the **dimension** of the vector space. A particular collection that spans the space and has the same number of vectors as the dimension of the space is called a **basis** for the vector space. Basis vectors (minimum number spanning set of vectors) have the special property that every vector in the space is **represented uniquely** by the collection of coefficients used to express it as a linear combination of the basis vectors. This property of unique representation in terms of the complex number coefficients in a linear combination is what allows us to represent vectors as lists of numbers and to do general quantitative manipulations/ calculations on vectors by manipulating collections of numbers. In our matrix arithmetic the scalar components of a ket are just the coefficients in an expansion of |v> as a linear combination of the basis vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ **.**

Choosing a basis to use to expand all the vectors is often called choosing a *representation* for the vector space. Our ability to choose representations is especially important when we want to create concrete computational models involving vectors of all kinds and to parameterize models in ways that make some calculations simple and some interesting properties readily apparent. We will most often choose a special set of basis vectors with additional geometric properties relative to an *Hermitian inner product* defined on the vector space. But I am getting ahead of myself. If you want to learn more about ramifications of ‘linear independence’, please be my guest and google “linear independence”; there is a Khan Academy lecture that covers the concept in the real case where geometric intuition is easily available.

Bra Vectors

Now we introduce another vector space that is distinct from, but closely related to, the space of ket vectors. Dirac called the vectors in this space **'bra' vectors. To distinguish them from kets and build a coherent notation, Dirac** introduced the abstract notation for a bra as : < name | (as compared to | name> for kets). *In our matrix arithmetic we represent bra vectors as rows instead of the column vectors (which are reserved for ket vectors).* In our concrete two dimensional complex world: a bra named v is represented as <v| = (a, b); where a and b are complex numbers. As with ket vectors, we can multiply bra vectors by complex numbers to produce new bra vectors and also add bra vectors together to make new ones. Generally we can form linear combinations of bra vectors to make new ones in the same way as with kets. All the same comments about span and basis also apply. BUT there is no sensible way we can add bras to kets. On the other hand we do have two ways to multiply kets and bras together by *matrix multiplication*. To be concrete, given a bra <v| =(a, b) and a ket |w>= $\begin{pmatrix} c \\ d \end{pmatrix}$, we can multiply them together to obtain a complex number (NOT another vector). The order is important:

<v| |w> = (a, b) $\begin{pmatrix} c \\ d \end{pmatrix}$:= ac+bd   [a complex number result]

This is the origin of the 'bra-ket' names, when they are multiplied in the particular bra-ket order a complex number is the result we get a 'bracket". Usually we don't bother with one of the two "|" symbols from the bra and ket and just write : < v | w>.

We can also matrix multiply bras and kets in the ket x bra order, but *the result is not a complex number*; instead we get a 2x2 matrix:

|w><v| = $\begin{pmatrix} c \\ d \end{pmatrix}$ (a, b) := $\begin{pmatrix} c\,a & c\,b \\ d\,a & d\,b \end{pmatrix}$ [a 2 x 2 matrix of complex numbers result]

One more time: multiplying a ket times a bra in this order creates a 2x2 matrix (which we will call an operator in this context) NOT another bra or ket or complex number.

Operator Matrices

You have just seen the last kind of object in our matrix arithmetic, 'square' ('square' in this context means the number of rows = number of columns so they actually look square) matrices of the dimension of the vector space; in our case 2 by 2. I will generally refer to these square matrices as "operators" in these notes.

Like each of the other elements (complex numbers, bras, kets) we can multiply operators by complex numbers and get another operator and we can also add operators together to make another operator, so we can make linear combinations of operators to make more operators. This means that operators, like bras and kets, form their own distinct vector space over the complex numbers. Unlike ket and bra vectors, we can multiply operators together to make another operator; that is, operators close under the operation of multiplication. In this way operators are like complex numbers. Unlike complex numbers, every non zero operator does NOT have an inverse under multiplication and operators generally have a rich algebraic structure depending strongly on their dimension. For 2x2 operators we summarize the addition and multiplication rules between operators:

Linear Combinations of Operators

If O1 and O2 are operators O1:=$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ O2:=$\begin{pmatrix} e & f \\ g & h \end{pmatrix}$ and α a complex number then

$$\alpha\, O1 = O1\, \alpha := \begin{pmatrix} \alpha\, a & \alpha\, b \\ \alpha\, c & \alpha\, d \end{pmatrix}, \quad O1 + O2 := \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} := \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}$$

Multiplication of Operators

The multiplication rule for operators generalizes the matrix multiplication for bras and kets, as it too proceeds 'row by column':

$$O1\, O2 := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} := \begin{pmatrix} \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} e \\ g \end{pmatrix} & \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} f \\ h \end{pmatrix} \\ \begin{pmatrix} c & d \end{pmatrix} \begin{pmatrix} e \\ g \end{pmatrix} & \begin{pmatrix} c & d \end{pmatrix} \begin{pmatrix} f \\ h \end{pmatrix} \end{pmatrix} = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix}$$

You can check from the formula above that generally O1 O2 $\neq$O2 O1. For any multiplication rule, if we get the same thing when multiplying in either order then we say that the operands commute. If the operands always commute , no matter what they are, then we say that the multiplication is commutative.(e.g. Multiplication of complex numbers is commutative. *Matrix* multiplication is generally not commutative though sometimes two particular square matrices may happen to commute.) It turns out to be useful to characterize the *failure* of two operators to commute (the failure to produce the same result when multiplied in different orders) with another operator, **the commutator of two operators, denoted by a square bracket notation:**

Commutator of O1 and O2 := [O1, O2] := O1 O2 - O2 O1

Obviously the commutator is zero if the two operators commute and non-zero other wise. Equally obvious is that every operator commutes with itself. A great deal of the essential information about the relationship between two different operators or amongst the operators in a larger collection can be deduced from the value of their commutator(s). In important contexts in Quantum Mechanics, the value of commutators are THE defining relationships within a system of operators.

There is much to say about the significance of commutators in quantum mechanics where we will represent things we can measure about a physical system using *Hermitian* operators. Commuting operators can represent compatible measurable quantities, whereas measurable quantities that cannot be simultaneously measured will have non-zero commutators. There is much more or say about commutators but that will have to wait. The name "operator" itself comes from the fact that operators can be multiplied with kets and bras to produce new kets and bras.

Multiplication of Operators with Kets and Bras

Given any operator O:=$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we can multiply it *on the left* with any ket and obtain a new ket:

O |v> = $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} v^1 \\ v^2 \end{pmatrix} = \begin{pmatrix} a\,v^1 + b\,v^2 \\ c\,v^1 + d\,v^2 \end{pmatrix}$

We can also multiply it *on the right* with any bra and get a new bra:

<w| O = $\begin{pmatrix} w^1 & w^2 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} w^1\,a + w^2\,c\;, & w^1\,b + w^2\,d \end{pmatrix}$

Hence operators define special kinds of functions on the space of kets in which the formula describing the map is that of an operator acting on a ket by matrix multiplication.

So **the order of allowed matrix multiplication is important in general**. Bra times ket is number, ket times bra is operator, operator times ket is ket, and bra times operator is bra. Generally when we get things from a multiplication procedure that depends on the order of the operands than we say the multiplication IS NOT commutative. **Matrix multiplication is not generally commutative** (though particular matrices may happen to commute). However, **matrix multiplication IS associative**. So in a triple product like bra × ket ×bra, we are free to compute the bra x ket product first to produce a number and then multiply that number by the final bra to get another bra in the end (i.e. bra×ket×bra=(bra × ket) ×bra = (complex number)× bra = final bra) OR we could multiply the final ket and bra to make an operator and then multiply the bra by the operator to get the final bra expression (i.e.bra×ket×bra= bra × (ket ×bra)= bra× operator = final bra). Either way the result is the same.

If you haven't much experience with manipulating matrices then this is already a lot to take in. The good news is that these are all the basic vector entities we will need to build our first quantum models. You need to become a master manipulator of 2 -D complex numbers, bras, kets, and operators. Generally you can form and manipulate linear combinations of like-kind objects (i.e.complex numbers with complex numbers, bras with bras, kets with kets, and operators with operators) and you can multiply different kinds of objects together to obtain another one of the four kinds of objects though some orders of multiplications are not sensible in this context. Multiplication, where allowed, is associative. Multiplication closes on the two sub-spaces of complex numbers and operators separately ; that is, in these two cases you can multiply like kind things and get another one of the same kind. (The same cannot be said for bras or kets, where only bra x ket and ket x bra make sense and which yield respectively a complex number and an operator.)

Summary of Products

The table below summarizes allowed products in our matrix arithmetic. In the table 'c number' means complex number. In the table below the result of the multiplication of the operand type in the left column is multiplied ***on the left*** to the operand in the intersecting column and the result is displayed. As noted in the text above, not all products "make sense"; however, when multiplying more than two matrices together, all possible sub products formed by associations of adjacent terms in the product are well defined. So this matrix multiplication closes on collections of the four types of objects (c numbers, bras, kets, and operators). All students in the Electrodynamics and Quantum Class are expected to be able to carry out these kinds of multiplications by hand.

Times	c number β	bra = (a, b)	ket = $\begin{pmatrix} c \\ d \end{pmatrix}$	op = $\begin{pmatrix} e & f \\ g & h \end{pmatrix}$
c number α	c number = $\alpha\beta$	bra = $(\alpha a, \alpha b)$	ket = $\begin{pmatrix} \alpha c \\ \alpha d \end{pmatrix}$	op = $\begin{pmatrix} \alpha e & \alpha f \\ \alpha g & \alpha h \end{pmatrix}$
bra = (u, v)	bra = $(\beta u, \beta v)$	NOT ALLOWED	c number = $u c + v d$	bra = $(u \;\; v) \begin{pmatrix} e & f \\ g & h \end{pmatrix} = (u e + v g, \;\; u f + v h)$
ket = $\begin{pmatrix} w \\ x \end{pmatrix}$	ket = $\begin{pmatrix} \beta w \\ \beta x \end{pmatrix}$	op = $\begin{pmatrix} wa & wb \\ xa & xb \end{pmatrix}$	NOT ALLOWED	NOT ALLOWED
op = $\begin{pmatrix} m & n \\ o & p \end{pmatrix}$	op = $\begin{pmatrix} \beta m & \beta n \\ \beta o & \beta p \end{pmatrix}$	NOT ALLOWED	ket = $\begin{pmatrix} m & n \\ o & p \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} mc + nd \\ oc + pd \end{pmatrix}$	op = $\begin{pmatrix} m & n \\ o & p \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} me + ng & mf + nh \\ oe + pg & of + ph \end{pmatrix}$

At this stage, I think its useful to visualize some typical forms of these row by column matrix multiplications. In what follows $\square$ represents *a* generic complex number.

$\square \; \square = \square$
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$\begin{pmatrix} \square \\ \square \end{pmatrix} \square = \begin{pmatrix} \square \\ \square \end{pmatrix}$
$(\square \;\; \square) \begin{pmatrix} \square \\ \square \end{pmatrix} = \square$
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$(\square \;\; \square) \begin{pmatrix} \square \\ \square \end{pmatrix} (\square \;\; \square) = (\square \;\; \square)$

Hermitian Conjugation

As noted earlier, ket and bra vectors are not the same animal (e.g. you can' t add them together or make a linear combination to make a new one) but they are closely related by a separate construction: Hermitian conjugation. In our matrix arithmetic we represent kets as columns of complex numbers and bras as rows of complex numbers. However, we will heavily employ an operation that allows us to make a bra from a ket and visa versa. That operation is called **Hermitian conjugation** and it is a generalization of complex conjugation for complex numbers (which we regard in this context as a special case). The easiest thing to do as this point is tell you how to compute the Hermitian Conjugate of each kind of object. After that, I will describe the operation in a few words and point out a few facts that follow from the definition. I will stay in the concrete 2-D world, but the idea easily extends to vector spaces and operators in any number of dimensions. We will denote the Hermitian conjugate generally by a superscript 'dagger', so that generally the Hermitian conjugate of A will be denoted by $A^\dagger$.

Object	Hermitian Conjugate
complex number α	$\alpha\dagger := \bar{\alpha}$ (complex conjugate of α)
ket $\mid v >: = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$	$\mid v > \dagger =$ bra $= < v \mid = \begin{pmatrix} \bar{\alpha} & \bar{\beta} \end{pmatrix}$
bra $< w \mid := (\alpha, \beta)$	$< w \mid \dagger =$ ket $= \mid w >= \begin{pmatrix} \bar{\alpha} \\ \bar{\beta} \end{pmatrix}$
operator $= O = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$	$O\dagger = \begin{pmatrix} \bar{\alpha} & \bar{\gamma} \\ \bar{\beta} & \bar{\delta} \end{pmatrix}$

Looking at the table, its easiest to describe the operation as the composition of matrix transposition (switching rows and columns) and complex conjugation of complex numbers. Transposition of a matrix leave a scalar (complex number) unchanged, turns a row vector into a column vector, a column into a row, and flips the elements of a square matrix symmetrically across the diagonal. To make the Hermitian Conjugate, after transposing the matrix, complex conjugate all the complex numbers. (Note: If all the numbers in our matrices were real numbers then Hermitian conjugation would amount to matrix transposition.)

At this point I need to point out something about the naming convention of bras and kets. If I have *defined* a ket |v> then what I mean by <v| is the Hermitian Conjugate of |v>; that is , <v| = $(\mid v >)^\dagger$. This is the typical order, defining vectors in ket representation and using the same symbol to label the corresponding bra obtained under Hermitian conjugation. On the rare occasion we will go the other way, and *define* the bra and then refer to the corresponding ket with the same label inside the |...> . Get used to this naming convention.

Basic Properties of Hermitian Conjugates of Linear Combinations and Products:

Since linear combinations of like objects (be they complex numbers, bras, kets, or operators) make sense, as do many products, there are three general rules worth knowing about the Hermitian conjugates of linear combinations and products:

$$(\alpha A + \beta B + ... + \omega Z)^\dagger = \bar{\alpha} A^\dagger + \bar{\beta} B^\dagger + ... + \bar{\omega} Z^\dagger$$

$\left(A^{\dagger}\right)^{\dagger} = A$

$(A\,B \ldots Z)^{\dagger} = Z^{\dagger} \ldots . B^{\dagger}\, A^{\dagger}$

Note carefully the reversal of the order of factors in the Hermitian conjugate of the product of matrices. The reversal of order is a consequence of the effect of transposition on the product of matrices.

SSince Hermitian conjugation turns bras into kets and kets into bras, there is no sensible way for the Hermitian conjugate of a bra or a ket to be equal to itself. **Not so for complex numbers or operators** (2 x 2 matrices). Complex numbers and operators can be 'self conjugate'. If a complex number is equal to its Hermitian conjugate then it is a real number! If we think of all the complex numbers as parametrized by two real numbers (the real and imaginary parts) then restricting to the complex numbers that are self conjugate knocks down the number of real number parameters by half. Viewed this way, the same sort of thing happens if we restrict the set of operators (2 x 2 matrices for us) to those that are equal to their Hermitian conjugates. The set of all operators can be parametrized by 2 x 4 = 8 real numbers. **If an operator is equal to its Hermitian conjugate then it is called an Hermitian operator** and just the like the complex number case, the set of Hermitian operators is 'half as big' as the set of all operators. The set of all Hermitian operators can be parametrized by 4 real numbers. Its also easy to see that every operator can be decomposed into a sum of Hermitian and anti-Hermitian parts; where anti Hermitian means $O^{\dagger} = -O$. (Just write $O = \frac{1}{2}(O + O^{\dagger}) + \frac{1}{2}(O - O^{\dagger})$, and check that the first term is Hermitian and the second is anti Hermitian).

We will have a great deal more to say about Hermitian operators and their important role in quantum mechanics. Hermitian operators have just the right properties to allow us to represent measurable quantities as Hermitian operators on the ket space of quantum states of a system. There is even a persistent debate whether or not EVERY Hermitian operator in a quantum mechanical model should be regarded as a measurable quantity. You will discover a number of these properties yourself in the exercises. Its easy to show that 2-D *Hermitian* operators can all be written in the form:

$$h = \begin{pmatrix} a & \alpha \\ \bar{\alpha} & b \end{pmatrix} \;;$$

where ***a and b are real numbers*** and ***α is a complex number***. Note that the count of real parameters is as advertised, with two real numbers on the diagonal and two real numbers in the complex number α. It is also useful to express Hermitian operators in 2-D as linear combinations **with real coefficients** of a standard set of 4 Hermitian matrices; these are the identity matrix ,$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$,and the three Pauli Matrices. The exercises will familiarize you with the Pauli Matrices. Feel free to google 'Pauli Matrices' to learn more about them. In the end, you will need to know everything there is to know about Pauli matrices.

Hermitian Inner Product

I am guessing that when you first learned about 'good old' real vectors, say to describe positions of points in a plane or 3D space relative to some fixed origin, you learned that vectors had two *geometric* properties: *magnitude* and *direction*. If you learned carefully, you understood that the notion of direction of a vector could be encoded into a dimensionless unit vector and the angles between vectors could be used to describe their relative directions. As you learned a little more, you were introduced to the notion of a 'dot' or 'inner' product that was defined in terms of the magnitudes and angle between two vectors OR you first worked in a Cartesian representation of vectors and defined the dot product as the sum of the products of the Cartesian components. If you did the later first, then hopefully you went back and proved that after defining the magnitude of a vector to be the square root of the dot product of a vector with itself that you could define the corresponding direction (unit vector) of a non-zero vector as the vector divided by its magnitude; and further define the angle between two (non zero) vectors as the inverse cosine of the dot product of the vectors divided by the product of their magnitudes. Regardless of the sequence in which you learned these things about the geometry of vectors in euclidean space (indeed, how vectors can be used to completely describe geometric relationships in this space) ,hopefully you came away with the correct impression that the dot product as a function on pairs of vector "knows" or encodes the basic geometry of 'arrow' vectors in two or three dimensional euclidean space. Here is the good news: The Hermitian inner product plays the same formal role for complex vectors as the dot product did for real vectors. In fact, if the components of our complex vectors happen to be real then the formula for the Hermitian inner product will immediately degenerate to the usual Cartesian formula for the dot product that you are used to from the real case.

But be warned, we aren't in Kansas (real euclidean vector spaces) any more, we are working with complex vectors and while many formal properties of complex vectors are the same as real vectors, the precise geometric interpretation of the inner or Hermitian product is more complex (that is a pun, get it?) and requires clarity and care so as to avoid making tacit assumptions that are true for real vectors but not for complex vectors.

Given what we have done so far, in particular having defined the Hermitian conjugate of a ket vector, we can define the Hermitian inner product between two ket vectors |w> and |v>, as a complex number valued function h(|w>, |v>) :

h(|w>, |v>) := <w | v >

[Be warned that the order of the kets is important and that some authors have made an abstractly equivalent but technically different definition which reverses the order of vector operands in this notation. Unfortunately, last time I checked, wikipedia authors have made this unfortunate choice.]

In our concrete 2-D world, if |w>= $\begin{pmatrix} w^1 \\ w^2 \end{pmatrix}$ and |v> = $\begin{pmatrix} v^1 \\ v^2 \end{pmatrix}$ then

$$h(|w>,|v>) = \langle w \mid v \rangle = \begin{pmatrix} \bar{w}^1 & \bar{w}^2 \end{pmatrix} \begin{pmatrix} v^1 \\ v^2 \end{pmatrix} = \bar{w}^1 v^1 + \bar{w}^2 v^2$$

Note carefully the complex conjugates of the w scalar components. This complex conjugation of the components of the first term in the Hermitian inner product means that the product is NOT a symmetric function of its two vector arguments: h(|w>, |v>) $\neq$ h(|v>, |w>); instead we have the following: h(|w>, |v>) = COMPLEX CONJUGATE of h(|v>, |w>) = $\overline{h(|\, v >,\ |\, w >)}$. When we reverse the order of the arguments we get the complex conjugate.

The following properties are easily seen to be true by a straightforward calculation from the definition given above

$h(\,|v>, \,|v>) = \bar{v}^1 v^1 + \bar{v}^2 v^2 = |v^1|^2 + |v^2|^2$ is a real number ≥ 0, $= 0$ iff $|v>$ is the zero vector
$h(\alpha\,|v> + \beta\,|w>, \,|x>) = \bar{\alpha}\, h(\,|v>, \,|x>) + \bar{\beta}\, h(\,|w>, \,|x>)$ note the conjugates of complex numbers α and β
$h(\,|x>, \,\alpha\,|v> + \beta\,|w>) = \alpha\, h(\,|x>, \,|v>) + \beta\, h(\,|x>, \,|w>)$

We could have done things in a different order. Had we defined the Hermitian inner between kets first, we could have gone back and defined a new vector space of bras using the Hermitian inner product. Since the emphasis of these notes is on the simplest and fastest route to the 2 dimensional arithmetic of QuBits, we have chosen the less strenuous algebraic path.

Mandatory Exercises: Manipulating Kets and Bras

For a general review of the operations involving matrices, (unfortunately mostly in real case), you might go through the Khan Academy treatment: https://www.khanacademy.org/math/algebra2/algebra-matrices

Problem #1

$$|u\rangle := \begin{pmatrix} 1 \\ 2i \end{pmatrix} \quad , \; |v\rangle = \begin{pmatrix} 2i \\ 3+4i \end{pmatrix}$$

Compute each of the following as an explicit complex number, column vector, row vector, or 2 x2 matrix, whichever is appropriate :

$$2i\,|u\rangle + (1+i)\,|v\rangle$$
$$\langle u|$$
$$\langle v|$$
$$\langle u|u\rangle$$
$$\langle v|v\rangle$$
$$\langle v|u\rangle$$
$$\langle u|v\rangle$$
$$|u\rangle\langle u|$$
$$|v\rangle\langle v|$$
$$P := \frac{|u\rangle\langle u|}{\langle u|u\rangle}$$

Problem #2

Prove that the operator P defined above satisfies the following conditions:

1) P^2 = P (Operators that satisfy this condition are called projection operators)

2) P |u> = |u>

3) P is Hermitian (i.e. $P^\dagger$ = P)

4) Prove that the statements 1-3 above are true for any |u> as long as |u> is a non-zero vector.

Problem #3

Suppose |w> = $\begin{pmatrix} a \\ b \end{pmatrix}$ and |up>=$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and |down>=$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

1) Prove that a=<up|w> and b=<down|w>.

2) Prove that |w>=a |up> + b|down>

3) Use your last result to prove that |w>=|up>(up|w>)+|down>(<down|w>);

4) Use your last result to prove that |w>= (|up><up|)|w>)+(|down><down|)|w>)

5) Show by direct computation that |up><up| + |down><down| = $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Problem #4

Suppose the |e1> and |e2> are two any non-zero kets with <e1|e2>=0.

Define the two projection operators P1= $\frac{|e1\rangle\langle e1|}{\langle e1|e2\rangle}$ and P2= $\frac{|e2\rangle\langle e2|}{\langle e2|e2\rangle}$.

Show that

1) P1+ P2 = $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (=:Identity Operator)

2) P1 P2 = P2 P1 = 0

Problem #5

1) Two kets, |u> and |v> are called linearly *independent* if the ONLY complex numbers α and β that solve the equation α |u> + β |v> = zero vector are BOTH α and β equal to zero. If two vectors are not linearly independent then they are called linearly *dependent*. Prove that two non-zero vectors that are linearly dependent are proportional by a complex number. [This is not hard, we are looking for clean logic!]

2) Two kets are called orthogonal if their Hermitian inner product is zero (<u|v> =0). Show that two non-zero orthogonal kets are also linearly independent.

Problem #6

The 3 Pauli Matrices are

$\sigma_x := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, and $\sigma_z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Prove by direct calculation that the Pauli Matrices have the following properties:

1) The Pauli matrices are all Hermitian and i times every Pauli matrix is anti-Hermitian

2) They have the following multiplication table (where 1 in the table represents the identity matrix = $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$)

Times	σ_x	σ_y	σ_z
σ_x	1	$i\,\sigma_z$	$-i\,\sigma_y$
σ_y	$-i\,\sigma_z$	1	$i\,\sigma_x$
σ_z	$i\,\sigma_y$	$-i\sigma_x$	1

3) We DEFINE the exponential function for matrices m by the infinite matrix series

Exp [m] = $1^{op} + m + \frac{1}{2} m^2 + ... + \frac{1}{n!} m^n + ...$

where 1^{op} is the identity matrix. Use this definition to show that Exp[i θ σ] = Cos[θ] 1^{op} + i Sin[θ] σ ; where θ is any **real** number and σ is any Pauli matrix. [Hint: what is the square of the matrix i σ, when σ is any one of the Pauli matrices ?]

4) Suppose we make an operator by adding together Pauli Matrices with **real** coefficients p^x, p^y, and p^z:

$P = p^x \sigma_x + p^y \sigma_y + p^z \sigma_z$

i) Show that P is an Hermitian operator

ii Compute P^2 and state the condition on the coefficients that guarantees that $P^2 = 1^{op}$

iii) Compute the determinant of P

Problem #7

i) Show that ANY 2x2 **Hermitian** Operator can be written as a unique linear combination of Pauli Matrices and the Identity operator with REAL coefficients , i.e., h $=p^t\, 1^{op} + p^x \sigma_x + p^y \sigma_y + p^z \sigma_z$; where p^t, p^x, p^y, p^z are real numbers.
ii) Find the expression for the determinant of any 2x2 hermitian matrix, h, expressed in terms of these four real variables defined above (p^t, p^x, p^y, p^z). [Recall: The determinant of any 2 x 2 matrix, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, is the scalar ad-bc.

Its common to write it this way det[$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$] =ad-bc.]
iii) Show that ANY 2x2 **Anti-Hermitian** Operator can be written as a unique linear combination of Pauli Matrices and the Identity operator with pure imaginary coefficients , i.e., h $=p^t\, 1^{op} + p^x \sigma_x + p^y \sigma_y + p^z \sigma_z$; where p^t, p^x, p^y, p^z are purely imaginary numbers.

iv) Show that ANY 2x2 operator what so ever can be written as a unique linear combination of Pauli Matrices and the Identity operator with complex coefficients , i.e., h $=p^t\, 1^{op} + p^x \sigma_x + p^y \sigma_y + p^z \sigma_z$; where p^t, p^x, p^y, p^z are complex numbers.

Problem #8

The inverse of a matrix M, denoted M^{-1}, is another matrix satisfying the matrix equation

M $M^{-1} = 1^{op}$.

i) Given a generic *2x2 matrix*, M=$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where a,b,c, and d are complex numbers, derive a formula for the matrix elements of $M^{-1} = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$ (that is, find formulas for x, y,z, and w in terms of a,b,c,and d by solving the system of 4 scalar equations component equations of the matrix equation $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$). Your result should only make sense if det [M] $\neq 0$.

ii) Check your formula obtained in part i) using the Pauli Matrices, since they are their own inverses!

iii) An operator is called *Unitary* if it's inverse is equal to its hermitian conjugate.
Show that the operator $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is Unitary if and only if the two kets |u>:=$\begin{pmatrix} a \\ c \end{pmatrix}$ and |v>:=$\begin{pmatrix} b \\ d \end{pmatrix}$ are orthogonal (i.e. <u|v>=<v|u>=0) AND of unit magnitude (i.e. <u|u>=<v|v> =1).